A pseudo–conformal representation of the Virasoro algebra

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Abstract

Generalizing the concept of primary fields, we find a new representation of the Virasoro algebra, which we call it a pseudo–conformal representation. In special cases, this representation reduces to ordinary– or logarithmic–conformal field theory. There are, however, other cases in which the Green functions differ from those of ordinary– or logarithmic–conformal field theories. This representation is parametrized by two matrices. We classify these two matrices, and calculate some of the correlators for a simple example.

In an ordinary conformal field theory primary fields are the highest weights of the representations of the Virasoro algebra. A primary field $\phi(w,\bar{w})$ can be defined through its operator product expansion with the stress-energy tensor $T(z)$ (and $\bar{T}(\bar{z})$) or equivalently through its commutation relations with the Laurent expansion coefficients of T; L_n's [1]:

$$[L_n, \phi_i(z)] = z^{n+1}\partial_z\phi_i + (n+1)z^n\Delta_i\phi_i, \tag{1}$$

where Δ_i is the conformal weight of ϕ_i. One can regard Δ_i's as the diagonal elements of a diagonal matrix D,

$$[L_n, \phi_i(z)] = z^{n+1}\partial_z\phi_i + (n+1)z^n D_{ij}\phi_j. \tag{2}$$

One can however, extend the above relation for any matrix D, which is not necessarily diagonal. This new representation of L_n also satisfies the Virasoro algebra for any arbitrary matrix D [2]. By a suitable change of basis, one can make D diagonal or Jordanian. If it is diagonalizable, the field theory is nothing but the ordinary conformal field theory (CFT). Otherwise it should be in the Jordanian form. The latter case is the logarithmic conformal field theory (LCFT) [3, 4, 2]. In the simplest case, the Jordanian block is two dimensional and the relation (2) for the two fields ϕ and ψ, becomes

$$\begin{aligned} &[L_n, \phi(z)] = z^{n+1}\partial_z\phi + (n+1)z^n\Delta\phi \\ &[L_n, \psi(z)] = z^{n+1}\partial_z\psi + (n+1)z^n\Delta\psi + (n+1)z^n\phi. \end{aligned} \tag{3}$$

The field ϕ is an ordinary primary field, and the field ψ is called a quasi-primary or logarithmic field and they transform in the following way:

$$\begin{aligned} &\phi(z) \to (\tfrac{\partial f^{-1}}{\partial z})^\Delta \phi(f^{-1}(z)) \\ &\psi(z) \to (\tfrac{\partial f^{-1}}{\partial z})^\Delta [\psi(f^{-1}(z)) + \log(\tfrac{\partial f^{-1}(z)}{\partial z})\phi(f^{-1}(z)] \end{aligned} \tag{4}$$

The two-point functions of these fields has been obtained in [3, 4]. It has been shown in [2] that any $n-$point function (for $n > 2$) containing the field ψ can be obtained through the $n-$point function containing the field ϕ instead of ψ.

Now the natural question which may arise is that, "is it possible to generalize (2) such that L_n's are still a representation of Virasoro algebra?" To investigate this question, we consider the following generalization of equation (2)

$$[L_n, \phi_i(z)] = z^{n+1}B_{ij}\partial_z\phi_j + (n+1)z^n A_{ij}\phi_j + C_i, \tag{5}$$

and impose the condition that L_n's satisfy the Virasoro algebra:

$$[[L_n, L_m], \phi_i] = (n-m)[L_{m+n}, \phi_i]. \tag{6}$$

Now it is easy to see that the generalization (5) satisfies the Virasoro algebra provided that the matrices A, B and C satisfy the following relations:

$$B^2 + BA - AB = B \tag{7}$$

$$BA = A \tag{8}$$

$$C = 0 \tag{9}$$

Now we try to classify the solutions of A and B. The trivial solution is $B = 1$, which is nothing but CFT, when A is diagonizable, and LCFT, when A is not diagonizable. If A is an invertible matrix the only solution for B is $B = 1$. However for any CFT which contains identity or any other field with zero conformal weight, A is not invertible, and there exists a corresponding new theory with $B \neq 1$ for which L_n's satisfy the Virasoro algebra. This is obviously not a CFT any more, as the action of any diffeomorphism on a field contains a term $-\xi \cdot \partial\phi$, where ξ is the generator of the diffeomorphism. This

corresponds to $B = 1$. For this reason, we call this representation a *pseudo–conformal* representation. Using (7,8) we have

$$(B - A)(B - 1) = 0. \tag{10}$$

Multiplying both sides of the above equation from the left by B, and using (8), leads to

$$(B - 1)^2 B = 0. \tag{11}$$

This means that the eigenvalues of B is equal to one or zero. So one can take the matrix B in the block diagonal form, where the blocks should be one of the following cases:
i) zero matrix
ii) identity matrix
iii) two dimensional Jordanian blocks $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

We choose a basis in which the matrices A and B are in the following form:

$$A = \begin{pmatrix} A_1 & | & A_2 \\ \text{—} & \text{—} & \text{—} \\ A_3 & | & A_4 \end{pmatrix} \qquad B = \begin{pmatrix} B_1 & | & 0 \\ -- & -|- & -- \\ 0 & | & 0 \end{pmatrix} \tag{12}$$

Using (8) and (10) it can be shown that $A_2 = A_3 = A_4 = 0$ and

$$B_1 A_1 = A_1. \tag{13}$$

Now the matrix B_1 can be written in the following form:

$$B_1 = \begin{pmatrix} 1 & 1| & 0 \\ \_0\_ & \_1|\_ & \_0\_ \\ 0 & 0| & 1 \end{pmatrix}, \tag{14}$$

where 1 stands for the identity matrix and the dimension of the first block should be even ($2k$). So

$$B_1 - 1 = \sum_{i=1}^{k} e_{i,i+k}, \tag{15}$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ which together with (13) yields:

$$(A_1)_{m+k,n}\Theta_{mk} = 0 \qquad \text{where} \qquad \Theta_{mk} = \begin{cases} 1 & m \leq k \\ 0 & m > k \end{cases}, \tag{16}$$

So all the elements of the lines $k + 1$ to $2k$ of the matrix A_1 should be zero. Now it is easy to show that

$$(A_1 - 1)(B_1 - 1) = 0, \tag{17}$$

where we have used

$$(B_1 - 1)^2 = 0. \tag{18}$$

Substituting equation (15) in (17) results:

$$(A_1 - 1)_{m,n-k} = 0, \qquad\qquad k < n \leq 2k. \tag{19}$$

and combining (16) and (19), the matrix A takes the following form

$$A = \begin{pmatrix} 1 & A'_1 & A'_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & A'_3 & A'_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{20}$$

Finally we use a similarity transformation which does not change B, to put the matrix A in a simpler form:

$$A = \begin{pmatrix} 1 & 0 & A'_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & A'_3 & A'_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{21}$$

Now we are at the point to calculate the correlation functions of the field ϕ_i, which have the Virasoro symmetry,i.e.

$$< [L_n, \phi_i \phi_j \cdots] >= 0 \qquad n = 0, \pm 1. \tag{22}$$

In general A'_2, A'_3 and A'_4 are arbitrary matrices. As an example we calculate two- and three-point functions in a simple case of (21):

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{23}$$

where 1 stands for identity matrix. If we write the fields in a coloumn matrix:

$$\Phi = \begin{pmatrix} (\phi_i^1) \\ (\phi_i^2) \end{pmatrix} \tag{24}$$

the relation (5) becomes:

$$[L_n, \phi_i^1] = z^{n+1} \partial_z (\phi_i^1 + \phi_i^2) + (n+1) z^n \phi_i^1 \tag{25}$$

$$[L_n, \phi_i^2] = z^{n+1} \partial_z \phi_i^2 \tag{26}$$

Noting (26) we see that ϕ_i^2's are like ordinary primary fields, with zero conformal weight of a CFT, so

$$< \phi_i^2(z) \phi_j^2(w) >= c. \tag{27}$$

To calculate the other two-point functions, we use (25) and(26), which leads to

$$< \phi_i^1(z) \phi_j^2(w) >= 0 \tag{28}$$

$$< \phi_i^1(z) \phi_j^1(w) >= \frac{d}{(z-w)^2} \tag{29}$$

Surprisingly, the above two-point functions are the same as the two-point functions of a CFT, in which conformal weights of the fields are zero and one. Now consider the three-point functions. The simplest case is $< \phi_i^2(z_1) \phi_j^2(z_2) \phi_k^2(z_3) >$. With a reason similar to that of the two-point functions, it is equal to:

$$< \phi_i^2(z_1) \phi_j^2(z_2) \phi_k^2(z_3) >= c_{ijk} \tag{30}$$

Using (25) and (26), one can also obtain

$$< \phi_i^2(z_1) \phi_j^2(z_2) \phi_k^1(z_3) >= \frac{\alpha_{ijk}(z_1 - z_2)}{(z_1 - z_3)(z_3 - z_2)} \tag{31}$$

where c_{ijk} and α_{ijk} are some constants which depends on i, j and k. This is also similar to three-point function of the fields of weight zero and one in an ordinary CFT. The three-point function $< \phi_i^2(z_1) \phi_j^1(z_2) \phi_k^1(z_3) >$ can also be calculated, which results

$$< \phi_i^2(z_1) \phi_j^1(z_2) \phi_k^1(z_3) >= \frac{\alpha_{ijk} z_2 + \alpha_{ikj} z_3 + d_{ijk}}{(z_2 - z_3)^2} \tag{32}$$

where d_{ijk} is a constant. This three-point function is completely different from the conformal case, and its dependence on coordinates is not only through their differences (as in CFT's). This behaviour results

from existance of nontrivial B matrix in our algebra. The correlator $<\phi_i^1(z_1)\phi_j^1(z_2)\phi_k^1(z_3)>$ can be also calculated, and it is seen that it is similar to that of ordinary CFT.

Using the above symmetry alone, one cannot obtain n-point functions with $n>3$, because there are only three first order partial differential equations for a function of more than three variables. This is as the case of ordinary CFT. One can, however, restrict the function to an $n-3$ variable function, just as in the case of ordinary CFT, and this restriction differs from that of an ordinary CFT.

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